

Rate functions for random walks on Random conductance models and related topics

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Abstract

We consider laws of the iterated logarithm and the rate function for sample paths of random walks on random conductance models under the assumption that the random walks enjoy long time sub-Gaussian heat kernel estimates.

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1 Introduction

The random conductance model (RCM) is a pair of a graph and a family of non-negative random variables (random conductances) which are indexed by edges of the graph. The RCM includes various important examples such as the supercritical percolation cluster, whose random conductances are i.i.d. Bernoulli random variables. In the recent progress on the RCM, various asymptotic behaviors of random walks are obtained on a class of RCM such as invariance principle, functional CLT, local CLT and long time heat kernel estimates. Here is a partial list of examples of the RCM;

1. Uniform elliptic case [13],
2. The supercritical percolation cluster [3],
3. I.i.d. unbounded conductance bounded from below [5],
4. I.i.d. bounded conductance under some tail conditions near 0 [10],
5. The level sets of Gaussian free field and the random interlacements [26].

We refer [28], [8], [24] for the invariance principle for random walks on the supercritical percolation cluster, [6] for the local limit theorem for random walks on the supercritical percolation cluster, [1] for the invariance principle on general i.i.d. RCMs, [2] for the Gaussian heat kernel upper bound on the possibly degenerate RCMs. We also refer [9] and [21] for more details about the RCM.

In [23], we discussed the laws of the iterated logarithms (LILs) for discrete time random walks on a class of RCM under the assumption of long time heat kernel estimates. The aims of this paper are to establish the laws of the iterated logarithm and to describe the rate functions for the sample paths of continuous time random walks on the RCM.

The LILs describe the fluctuation of stochastic processes, which was originally obtained by Khinchin [18] for a random walk. We establish the LIL w.r.t. both $\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)$ and $d(Y_0^\omega, Y_t^\omega)$, and another LIL, which describes liminf behavior of $\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)$.

The rate function describes the sample path ranges of stochastic processes. For d -dimensional Brownian motion $B = \{B_t\}_{t \geq 0}$, the Kolmogorov test tells us that

$$\mathbb{P}\left(|B_t| \geq t^{1/2}h(t) \text{ for sufficiently large } t\right) = \begin{cases} 1 \\ 0, \end{cases} \quad \text{according as} \quad \int_1^\infty \frac{1}{t} h(t)^d e^{-\frac{h(t)^2}{2}} dt \begin{cases} < \infty \\ = \infty, \end{cases}$$

where $h(t)$ is a positive function such that $h(t) \nearrow \infty$ as $t \rightarrow \infty$. For $d \geq 3$, the Dvoretzky and Erdős test tells us that

$$\mathbb{P}\left(|B_t| \geq t^{1/2}h(t) \text{ for sufficiently large } t\right) = \begin{cases} 1 \\ 0, \end{cases} \quad \text{according as} \quad \int_1^\infty \frac{1}{t} h(t)^{d-2} dt \begin{cases} < \infty \\ = \infty, \end{cases} \quad (1.1)$$

where $h(t)$ is a positive function such that $h(t) \searrow 0$ as $t \rightarrow \infty$. These results were extended to various frameworks such as symmetric stable processes on \mathbb{R}^d , Brownian motions on Riemannian manifolds, symmetric Markov chains on weighted graphs and β stable like processes ($\beta \geq 2$). We establish an analogue of (1.1) w.r.t. random walks on the RCM.

Our approach is as follows; We assume quenched heat kernel estimates and establish both quenched LILs and an analogue of the Dvoretzky and Erdős test. As we will see in Section 1.2, our results are applicable for various models since heat kernel estimates are obtained for random walks on various RCMs. The concrete examples are given in Section 1.2.

The organization of this paper is as follows. First, we give the framework and main results of this paper in Section 1.1 and examples in Section 1.2. In Section 2 we establish some preliminary results. In Section 3 we give the proof of the LILs. In Section 4 we establish an analogue of (1.1). Finally in Section 5 we discuss the case where $G = \mathbb{Z}^d$ and the media is ergodic.

In this paper, we use the following notation.

- Notation.** (1) We use c, C, c_1, c_2, \dots as the deterministic positive constants. These constants do not depend on the random environment ω , time parameters t, s, \dots , distance parameters r, \dots , and vertices of graphs.
- (2) We define $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

1.1 Framework and Main results

Let $G = (V, E) = (V(G), E(G))$ be a countable and connected graph of bounded degree, i.e. $M := \sup_{x \in V(G)} \deg x < \infty$. We write $x \sim y$ if $(x, y) \in E(G)$. A sequence $\ell_{xy} : x = x_0, x_1, \dots, x_n = y$ on G is called a path from x to y if $x_i \sim x_{i+1}$ for all $i = 0, 1, \dots, n-1$. We write $d(\cdot, \cdot)$ as the usual graph distance, that is, the length of a shortest path in G , and denote $B(x, r) = \{y \in V(G) \mid d(x, y) \leq r\}$.

Throughout of this paper we assume that there exist $\alpha \geq 1$, $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \#B(x, r) \leq c_2 r^\alpha \quad (1.2)$$

for any $x \in V(G)$ and $r \geq 1$.

We introduce the random conductance model below. Let $\omega = \{\omega_e = \omega_{xy}\}_{e=(x,y) \in E(G)}$ be a family of non-negative weight which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For non-negative weights $\omega = \{\omega_e\}_e$, we define $\pi^\omega(x) = \sum_{y: y \sim x} \omega_{xy}$ and $\nu^\omega(x) = 1$. We fix a base point $x_0 \in V(G)$, and define graphs $G^\omega = (V(G^\omega), E(G^\omega))$ as

$$\begin{aligned} V(G^\omega) &= \left\{ y \in V(G) \mid \begin{array}{l} \text{There exists a path } \ell_{x_0 y} : x_0, x_1, \dots, x_n = y \text{ such that} \\ \omega_{x_i x_{i+1}} > 0 \text{ for all } i = 0, 1, \dots, n-1. \end{array} \right\}, \\ E(G^\omega) &= \{e = (x, y) \in E(G) \mid x, y \in V(G^\omega) \text{ and } \omega_{xy} > 0\}. \end{aligned}$$

We denote $d^\omega(\cdot, \cdot)$ as the graph distance of G^ω . Note that $G^\omega = G$ and $d^\omega = d$ if conductance ω is strictly positive.

We will consider two types of random walks, constant speed random walk (CSRW) and variable speed random walk (VSRW) associated to $\omega \in \Omega$. Both CSRW and VSRW are continuous time random walk whose transition probability is given by $P^\omega(x, y) = \frac{\omega_{xy}}{\pi^\omega(x)}$. For the CSRW, the holding time distribution at $x \in V(G^\omega)$ is $\text{Exp}(1)$, whereas for the VSRW, the holding time distribution at $x \in V(G^\omega)$ is $\text{Exp}(\pi^\omega(x))$. We write $\mathcal{L}_\theta^\omega$ for the generator which is given by

$$\mathcal{L}_\theta^\omega f(x) = \frac{1}{\theta \pi^\omega(x)} \sum_{y: y \sim x} (f(y) - f(x)) \omega_{xy},$$

and we also write the corresponding heat kernel as

$$q_t^\omega(x, y) = \frac{P^\omega(x, y)}{\theta^\omega(y)},$$

where $\theta^\omega = \pi^\omega$ for the CSRW case and $\theta^\omega \equiv 1$ for the VSRW case. We write $Y^\omega = \{Y_t^\omega\}_{t \geq 0}$ as either the CSRW or the VSRW, P_x^ω as the law of the random walk Y^ω which starts at x , and

$$\tau_F = \tau_F^\omega = \inf\{t \geq 0 \mid Y_t^\omega \notin F\}, \quad \sigma_F = \sigma_F^\omega = \inf\{t \geq 0 \mid Y_t^\omega \in F\}, \quad \sigma_F^+ = \sigma_F^{+\omega} = \inf\{t > 0 \mid Y_t^\omega \in F\}. \quad (1.3)$$

We denote $F^\omega = F \cap V(G^\omega)$, $V^\omega(F) = \sum_{y \in F \cap V(G^\omega)} \theta^\omega(y)$ for $F \subset V(G)$ and $V^\omega(x, r) = V^\omega(B(x, r))$. We write $T_0^\omega = 0$ and $T_{n+1}^\omega = \inf\{t > T_n^\omega \mid Y_t^\omega \neq Y_{T_n^\omega}^\omega\}$, and introduce a discrete time random walk $\{X_n^\omega := Y_{T_n^\omega}^\omega\}_{n \geq 0}$.

First, we state the results about LILs. To do this, we need the following assumptions.

Assumption 1.1. *There exist positive constants ϵ, β such that $\epsilon < \beta + 1$ and a family of non-negative random variables $\{N_x = N_{x, \epsilon}\}_{x \in V(G)}$ such that the following hold;*

(1) *There exist positive constants $c_{1.1}, c_{1.2}, c_{1.3}, c_{1.4}$ such that*

$$q_t^\omega(x, y) \leq \begin{cases} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left(-c_{1.2} \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right), & \text{if } t \geq d(x, y), \\ c_{1.3} \exp\left(-c_{1.4} d(x, y) \left(1 \vee \log \frac{d(x, y)}{t}\right)\right), & \text{if } t \leq d(x, y), \end{cases} \quad (1.4)$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ and $t \geq N_x(\omega)$.

(2) *There exist positive constants $c_{2.1}, c_{2.2}$ such that*

$$q_t^\omega(x, y) \geq \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left(-c_{2.2} \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right) \quad (1.5)$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ and $t \geq 0$ with $d(x, y)^{1+\epsilon} \vee N_x(\omega) \leq t$.

(3) *There exist positive constants $c_{3.1}, c_{3.2}$ such that*

$$c_{3.1} r^\alpha \leq V^\omega(x, r) \leq c_{3.2} r^\alpha \quad (1.6)$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$ and $r \geq N_x(\omega)$.

(4) *There exist positive constants $c_{4.1}, c_{4.2}, c_{4.3}, c_{4.4}, c_{4.5}$ such that*

$$q_t^\omega(x, y) \leq \begin{cases} \frac{c_{4.1}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp\left(-c_{4.2} \frac{d(x, y)^2}{t}\right), & \text{if } t \geq c_{4.3} d(x, y), \\ \frac{c_{4.4}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp\left(-c_{4.5} d(x, y) \left(1 \vee \log \frac{d(x, y)}{t}\right)\right), & \text{if } t \leq c_{4.3} d(x, y), \end{cases} \quad (1.7)$$

for almost all $\omega \in \Omega$, all $t > 0$ and $x, y \in V(G^\omega)$ with $d(x, y) \geq N_x(\omega) \wedge N_y(\omega)$.

(1.7) is called Carne-Varopoulos bound. Note that (1.4) holds for $t \geq N_x(\omega)$ while (1.7) holds for all $t > 0$. It is known that (1.7) holds under general conditions which will be described in the following Proposition (see [16, Theorem 2.1, 2.2]).

Proposition 1.2. *Let $\{N_x\}$ be as in Assumption 1.1 and $d_\theta^\omega(\cdot, \cdot)$ be a metric on $G^\omega = (V(G^\omega), E(G^\omega))$ which satisfies*

$$\frac{1}{\theta^\omega(x)} \sum_{y \in V(G^\omega)} d_\theta^\omega(x, y)^2 \omega_{xy} \leq 1. \quad (1.8)$$

If there exists a positive constant c such that $d_\theta^\omega(x, y) \geq cd(x, y)$ for all $x, y \in V(G^\omega)$ with $d(x, y) \geq N_x(\omega) \wedge N_y(\omega)$, then (1.7) holds.

Next we assume the following three types of integrability conditions.

Assumption 1.3. Let $\{N_x\}_{x \in V(G)}$ be as in Assumption 1.1 and define $f(t) = f_\epsilon(t) = \mathbb{P}(N_x \geq t)$. We consider the following three types of integrability conditions.

- (1) $\sum_{n \geq 1} n^\alpha f(n) < \infty$,
- (2) $\sum_{n \geq 1} n^{\alpha\beta} f(n) < \infty$,
- (3) For positive and non-increasing function $h(t)$, $\sum_n n^\alpha f(nh(n^\beta)) < \infty$.

We now state the main results of this paper.

Theorem 1.4. (1) Under Assumption 1.1 (1) (2) (3) and Assumption 1.3 (1), for almost all $\omega \in \Omega$ there exists positive numbers $c_1 = c_1^\omega, c_2 = c_2^\omega$ such that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta} (\log \log t)^{1-1/\beta}} &= c_1, & P_x^\omega \text{-a.s. for all } x \in V(G^\omega), \\ \limsup_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta} (\log \log t)^{1-1/\beta}} &= c_2, & P_x^\omega \text{-a.s. for all } x \in V(G^\omega). \end{aligned} \quad (1.9)$$

(2) Under Assumption 1.1 (1) (2) (3) and Assumption 1.3 (2), for almost all $\omega \in \Omega$ there exist a positive number $c_3 = c_3^\omega$ such that

$$\liminf_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta} (\log \log t)^{1-1/\beta}} = c_3, \quad P_x^\omega \text{-a.s. for all } x \in V(G^\omega). \quad (1.10)$$

Theorem 1.5. Suppose Assumption 1.1 (1) (2) (3) (4) and $\alpha/\beta > 1$. In addition $\theta^\omega(x) = \pi^\omega(x) \geq c$ for a positive constant $c > 0$ in the case of CSRW. Let $h : (1, \infty) \rightarrow (0, \infty)$ be a function such that $h(t) \searrow 0$ as $t \rightarrow \infty$ and the function $\varphi(t) := t^{1/\beta} h(t)$ is increasing. If $h(t)$ satisfies Assumption 1.3 (3) and

$$\int_1^\infty \frac{1}{t} h(t)^{\alpha-\beta} dt < \infty \text{ or } = \infty$$

then

$$P_x^\omega \left(d(x, Y_t^\omega) \geq t^{1/\beta} h(t) \text{ for all sufficiently large } t \right) = 1 \text{ or } 0.$$

Finally we discuss the constants c_1, c_2, c_3 in (1.9) and (1.10). When we consider a case of $G = \mathbb{Z}^d$, we can take c_1, c_2 as deterministic constants under some appropriate assumptions. To state this, we take the base point $x_0 = 0 \in \mathbb{Z}^d$ and we write shift operators as $\tau_x, (x \in \mathbb{Z}^d)$, where τ_x is given by

$$(\tau_x \omega)_{yz} = \omega_{x+y, x+z}. \quad (1.11)$$

We assume the following conditions.

Assumption 1.6. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies the following conditions;

- (1) \mathbb{P} is ergodic with respect to the translation operators τ_x , namely $\mathbb{P} \circ \tau_x = \mathbb{P}$ and if $\tau_x(A) = A$ for all $x \in \mathbb{Z}^d$ and for all $A \in \mathcal{F}$ then $\mathbb{P}(A) = 0$ or 1 .
- (2) For almost all environment ω , $V(G^\omega)$ contains a unique infinite connected component.
- (3) (VSRW case) $\mathbb{E} \left[\frac{1}{\pi^\omega(0)} \right] \in (0, \infty)$.

Theorem 1.7. Suppose that the same assumptions as in Theorem 1.4 and suppose in addition Assumption 1.6. Then we can take c_1, c_2, c_3 in (1.9) and (1.10) as deterministic constants (i.e. do not depend on ω).

1.2 Example

In this subsection, we give some examples for which our results are applicable.

Example 1.8 (Bernoulli supercritical percolation cluster). Let $G = (\mathbb{Z}^d, E_d)$ be a graph, where $E_d = \{\{x, y\} \mid x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$. Put a Bernoulli random variable ω_e with $\mathbb{P}(\eta_e = 1) = p$ on each edge. This model is called bond percolation. We write $p_c(d)$ as the critical probability. It is known that there exists a unique infinite connected component when $p > p_c(d)$. See [17] for more details about the percolation.

Barlow [3] proved that heat kernels of CSRWs on the super-critical percolation cluster (that is, when $p > p_c(d)$) on \mathbb{Z}^d , $d \geq 2$ satisfy Assumption 1.1 (1) (2) (3) (4), Assumption 1.3 (1) (2) with $\alpha = d$, $\beta = 2$ and $f_\epsilon(t) = c \exp(-c't^\delta)$ for some $c, c', \delta > 0$. Since the media is i.i.d. and there exists an unique infinite connected component, we can obtain Theorem 1.4 with deterministic constants by Theorem 1.7.

In addition, we can easily check that $h(t) = \frac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa > 0$ satisfy the conditions in Assumption 1.3 (3) and the assumptions of Theorem 1.5 in the case of $d > 2$. Thus $P_x^\omega(d(x, Y_t^\omega) \geq t^{1/\beta} h(t)) = 1, 0$ according as $\kappa > d - 2, \leq d - 2$ respectively by Theorem 1.5.

Note that (1.9) for the supercritical percolation cluster was already obtained by [15, Theorem 1.1].

Example 1.9 (Gaussian free fields and random interlacements). Gaussian free field on a graph $G = (V, E)$ is a family of centered Gaussian variables $\{\varphi_x\}_{x \in G}$ with covariance $E[\varphi_x \varphi_y] = g(x, y)$, where $g(x, y)$ is the Green function of a random walk on G . Here we are interested in the level sets of the Gaussian free field $E_h = \{x \in V \mid \varphi_x \geq h\}$. We can regard the level sets as one of the percolation models which has correlation among the vertices in V . See [30] for the details.

The random interlacements concern geometries of random walk trajectories, e.g. how many random walk trajectories are needed to make the underlying graph disconnected? Sznitman [29] formulated the model of random interlacements. However the model of random interlacements is defined through Poisson point process on a trajectory space, we can also regard this model as the percolation model with long range correlation. From the viewpoint of the RCM, we can regard the model of random interlacements as one of the RCM whose conductances take the value 0 or 1 and the conductances are not independent. See [14] for the details.

Sapozhnikov [26, Theorem 1.15] proved that for \mathbb{Z}^d , $d \geq 3$, the CSRWs on (i) certain level sets of Gaussian free fields; (ii) random interlacements at level $u > 0$; (iii) vacant sets of random interlacements for suitable level sets, satisfy our Assumption 1.1 (1) (2) (3) with $\alpha = d$, $\beta = 2$ and the tail estimates of $N_x(\omega)$ as $f_\epsilon(t) = c \exp(-c'(\log t)^{1+\delta})$ for some $c, c', \delta > 0$. As the same reason with the case of Bernoulli supercritical percolation cluster, Assumption 1.1 (3) is also satisfied in these models. This subexponential tail estimate is sufficient for Assumption 1.3 (3) with $h(t) = \frac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa > 0$. Since the media is ergodic and there is an unique infinite connected components (see [25], [29, Corollary 2.3] and [32, Theorem 1.1]), Theorem 1.4 holds with deterministic constants by Theorem 1.7, and Theorem 1.5 holds with $h(t) = \frac{1}{(\log t)^{\kappa/(d-2)}}$ for $\kappa \geq d - 2$ or $< d - 2$ respectively.

Example 1.10 (Uniform elliptic case). Suppose that a graph $G = (V, E)$ is endowed with weight 1 on each edge and satisfies (1.2) and the scaled Poincaré inequalities. Take c_1, c_2 as positive constants and put random conductances on all edges so that $c_1 \leq \omega(e) \leq c_2$ for all $e \in E$ and for almost all ω . Delmotte [13] obtained Gaussian heat kernel estimates for CSRWs in this framework. Thus Assumption 1.1 (1) (2) (3) hold with $\beta = 2$ and $N_{x, \epsilon} \equiv 1$. Hence Theorem 1.4 holds.

In addition, Assumption 1.1 is followed by [12, Corollary 11, 12]. (See also Proposition 1.2, note that the graph distance satisfies (1.8) for CSRW case.) Thus Theorem 1.5 holds with $h(t) = \frac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d - 2$ or $< d - 2$ respectively).

Example 1.11 (Unbounded conductance bounded from below). Let $G = \mathbb{Z}^d$ ($d \geq 2$) and put random conductances $\omega = \{\omega_{xy}\}_{xy \in E}$ which take the value $[1, \infty)$. Barlow and Deuschel [5, Theorem 1.2] proved that the heat kernels of VSRW satisfy Assumption 1.1 (1) (2), Assumption 1.3 (1) (2) with $\alpha = d$, $\beta = 2$ and $f_\epsilon(t) = c_1 \exp(-c_2 t^\delta)$ for some $c_1, c_2, \delta > 0$. (Note that Assumption 1.1 (3) is trivial since $V^\omega(x, r) = \sharp B(x, r)$ for the VSRW.) Hence Theorem 1.4 holds.

In addition, Assumption 1.1 (4) is followed by [5, Theorem 2.3, Theorem 4.3 (b)] or [16, Theorem 2.1, Theorem 2.2]. Thus Theorem 1.5 for the VSRW holds with $h(t) = \frac{1}{(\log t)^{\kappa/(d-2)}}$ ($\kappa \geq d-2$ or $< d-2$ respectively).

Moreover, if the conductances $\{\omega_e\}_e$ satisfy Assumption 1.6 (3) then Theorem 1.4 holds with deterministic constants.

2 Consequences of Assumption 1.1

In this section we give some preliminary results of our assumptions.

2.1 Consequences of heat kernel estimates

In this subsection, we give preliminary results of Assumption 1.1 (1) (2) (3).

Recall the notations in (1.4).

Lemma 2.1. *Suppose Assumption 1.1 (1) (3). For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$ there exist positive constants $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, $c_3 = c_3(\delta)$ such that*

$$P_x^\omega(d(x, Y_t^\omega) \geq r) \leq c_1 \exp \left[-(c_{1.2} - \delta) \left(\frac{r}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right] + c_2 \exp(-c_3 t) \quad (2.1)$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$, $r \geq N_x(\omega)$ and $t \geq N_x(\omega)$.

This lemma is standard except for the part of estimates of Poissonian regime (the bottom line of (1.4)). For completeness I give the proof here.

Proof. We first prepare some preliminary facts to estimate $P_x^\omega(d(x, Y_t^\omega) \geq r)$. Set $h_1(\eta, s) = \exp[-\eta s^{\beta/(\beta-1)}]$ and $h_2(\eta, s) = \exp[-\eta s]$. For $h_1(\eta, s)$, we can easily see that there exists a constant $\zeta_0 > 1$ such that

$$h_1(\eta, \zeta s) \leq h_1(\eta, 1) h_1(\eta, s) \quad (2.2)$$

for all $\zeta \geq \zeta_0$, $\eta > 0$ and $s \geq 1$. (We can take ζ_0 as the positive number which satisfies $\zeta_0^{\beta/(\beta-1)} - 1 = 1$.) For $h_2(\eta, s)$, we can easily see that

$$h_2(\eta, \zeta s) \leq h_2(\eta, 1) h_2(\eta, s) \quad (2.3)$$

for all $\zeta \geq 2$, $\eta > 0$ and $s \geq 1$. Next, we easily see that for all $\zeta > 1$ there exists $c_1 = c_1(\zeta)$ such that for almost all $\omega \in \Omega$

$$V^\omega(x, r\zeta) \leq c_1 V^\omega(x, r) \quad (2.4)$$

for all $x \in V(G)$ and for all $r \geq N_x(\omega)$. (Use (1.6) and take $c_1 = \frac{c_{3.2}\zeta^\alpha}{c_{3.1}}$.) Thirdly, it is also easy to see that for all $\delta \in (0, c_{1.2})$ there exists $c_2(\delta)$ such that

$$s^\alpha \exp[-c_{1.2} s^{\beta/(\beta-1)}] \leq c_2(\delta) \exp[-(c_{1.2} - \delta) s^{\beta/(\beta-1)}] \quad (2.5)$$

for all $s \geq 1$, where $c_{1.2}$ is the same constant as in (1.4). We can also see that for all $\delta \in (0, c_{1.4})$ there exists a positive constant $c_3 = c_3(\delta)$ such that

$$s^\alpha \exp[-c_{1.4} s] \leq c_3(\delta) \exp[-(c_{1.4} - \delta) s] \quad (2.6)$$

for all $s \geq 1$. Using (2.5), we can see that for $d(x, z) \geq s \geq t^{1/\beta}$ and $\delta \in (0, c_{1.2})$

$$\begin{aligned} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp \left[-c_{1.2} \left(\frac{d(x, z)}{t^{1/\beta}} \right)^{\beta/(\beta-1)} \right] &= \frac{c_{1.1}}{d(x, z)^\alpha} \left(\frac{d(x, z)}{t^{1/\beta}} \right)^\alpha \exp \left[-c_{1.2} \left(\frac{d(x, z)}{t^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \\ &\leq \frac{c_4(\delta)}{d(x, z)^\alpha} \exp \left[-(c_{1.2} - \delta) \left(\frac{d(x, z)}{t^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \quad (\text{use (2.5)}) \\ &\leq \frac{c_4(\delta)}{s^\alpha} \exp \left[-(c_{1.2} - \delta) \left(\frac{s}{t^{1/\beta}} \right)^{\beta/(\beta-1)} \right], \quad (\text{use } d(x, z) \geq s). \end{aligned} \quad (2.7)$$

Now we estimate $P_x^\omega(d(x, Y_t^\omega) \geq r)$. We first consider the case $r \leq t^{1/\beta}$. Since $s \mapsto h_1(\eta, s)$, ($\eta > 0$) is non-increasing, we have

$$P_x^\omega(d(x, Y_t^\omega) \geq r) \leq 1 \leq \frac{h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right)}{h_1(c_{1.2}, 1)} = c_5 h_1\left(c_{1.2}, \frac{r}{t^{1/\beta}}\right), \quad (2.8)$$

where we set $c_5 = 1/h(c_{1.2}, 1)$. So we may and do assume $r \geq t^{1/\beta}$. Take $\zeta \geq \zeta_0 \vee 2$ so that (2.2), (2.3) and (2.4) hold. We divide $P_x^\omega(d(x, Y_t^\omega) \geq r)$ into

$$\sum_{k=0}^K \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} q_t^\omega(x, z) \theta^\omega(z), \quad \sum_{k=K}^\infty \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} q_t^\omega(x, z) \theta^\omega(z), \quad (2.9)$$

where K is the positive integer which satisfies $r\zeta^K \leq t < r\zeta^{K+1}$. We have for $t \geq N_x(\omega)$, $r \geq N_x(\omega)$ and using (1.4)

$$\begin{aligned} (\text{The first term of (2.9)}) &\leq \sum_{k=0}^K \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2} \left(\frac{d(x, z)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \theta^\omega(z) \\ &\leq \sum_{k=0}^K \frac{c_6(\delta)}{(r\zeta^k)^\alpha} \exp\left[-(c_{1.2} - \delta) \left(\frac{r\zeta^k}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] (r\zeta^{k+1})^\alpha \quad (\text{use (2.7) and (1.6)}) \\ &\leq \sum_{k=0}^K c_7(\delta, \zeta) h_1\left(c_{1.2} - \delta, \frac{r\zeta^k}{t^{1/\beta}}\right) \\ &\leq c_7(\delta, \zeta) h_1\left(c_{1.2} - \delta, \frac{r}{t^{1/\beta}}\right) \sum_{k=0}^K h_1(c_{1.2} - \delta, 1)^k \quad (\text{use (2.2)}) \\ &\leq c_8(\delta, \zeta) \exp\left[-(c_{1.2} - \delta) \left(\frac{r}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right], \quad (\text{since } h_1(c_{1.2} - \delta, 1) < 1). \end{aligned} \quad (2.10)$$

For the second term of (2.9), using (1.4), $t \geq N_x(\omega)$ and $r \geq N_x(\omega)$ we have

$$\begin{aligned} (\text{The second term of (2.9)}) &\leq \sum_{k=K}^\infty \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} c_{1.3} \exp\left[-c_{1.4} d(x, z) \left(1 \vee \log \frac{d(x, z)}{t}\right)\right] \theta^\omega(z) \\ &\leq \sum_{k=K}^\infty \sum_{z \in B^\omega(x, r\zeta^{k+1}) \setminus B^\omega(x, r\zeta^k)} c_{1.3} \exp[-c_{1.4} d(x, z)] \theta^\omega(z) \quad \left(\text{since } 1 \vee \log \frac{d(x, z)}{t} \geq 1\right) \\ &\leq \sum_{k=K}^\infty c_9 \exp[-c_{1.4}(r\zeta^k)] (r\zeta^{k+1})^\alpha \quad (\text{use (1.6)}) \\ &\leq c_{10}(\zeta, \delta) \sum_{k=K}^\infty \exp[-(c_{1.4} - \delta)r\zeta^k] \quad (\text{use (2.6)}) \\ &= c_{10}(\zeta, \delta) \sum_{k=K}^\infty h_2(c_{1.4} - \delta, r\zeta^k) \\ &\leq c_{11}(\zeta, \delta) h_2(c_{1.4} - \delta, r\zeta^K) \sum_{k=0}^\infty h_2(c_{1.4} - \delta, 1)^k \quad (\text{use (2.3)}) \\ &\leq c_{12}(\zeta, \delta) \exp[-c_{13}(\zeta, \delta)t], \quad (\text{since } r\zeta^K \leq t < r\zeta^{K+1}). \end{aligned} \quad (2.11)$$

Therefore, by (2.8), (2.10), (2.11) and adjusting the constants, we obtain (2.1). We thus complete the proof. \square

Again recall the notations $c_{1.2}$ and $c_{1.4}$ in (1.4).

Lemma 2.2. Suppose Assumption 1.1 (1) (3). For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$ there exist positive constants $c_1 = c_1(\delta)$, $c_2 = c_2(\delta)$, $c_3 = c_3(\delta)$ such that

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \geq 2r \right) \leq c_1 \exp \left[-(c_{1.2} - \delta) \left(\frac{r}{(2t)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] + c_2 \exp[-c_3 t] \quad (2.12)$$

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(y, Y_s^\omega) \geq 4r \right) \leq c_1 \exp \left[-(c_{1.2} - \delta) \left(\frac{r}{(2t)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] + c_2 \exp[-c_3 t] \quad (2.13)$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$, $t \geq 1$ and $r \geq 1$ with $d(x, y) \leq 2r$, $t \geq \max_{u \in B(x, 2r)} N_u(\omega)$ and $r \geq \max_{u \in B(x, 2r)} N_u(\omega)$.

Proof. This is standard (see the proof of [4, Lemma 3.9 (c)]), so we omit the proof. \square

Lemma 2.3. Suppose Assumption 1.1 (1) (2) (3). Then there exist positive constants $\eta \geq 1$, $c_1, c_2 > 0$ such that

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \leq 3\eta r \right) \geq c_1 \exp \left[-c_2 \frac{t}{r^\beta} \right] \quad (2.14)$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$, $t \geq r \geq 1$ with $r^{1/\beta} \geq \max_{z \in B(y, 3\eta r)} N_z(\omega)$.

Proof. The proof is quite similar to that of [22, Proposition 3.3], so we omit the proof. \square

Let c_1, c_2 be as in Lemma 2.3. Note that we can assume that $c_1 < 1$ (and therefore $c_1 \exp[-c_2] \in (0, 1)$). We define $\rho_1, a_k, b_k, \lambda_k, u_k, \sigma_k$ as

$$\begin{aligned} \rho_1 &= c_1 \exp[-c_2], \quad a_k^\beta = e^{k^2}, \quad b_k^\beta = e^k, \\ \lambda_k &= \frac{2}{3|\log \rho_1|} \log(1+k), \quad u_k = \lambda_k a_k^\beta, \quad \sigma_k = \sum_{i=1}^{k-1} u_i. \end{aligned} \quad (2.15)$$

Corollary 2.4 (Corollary of Lemma 2.3). Let $\eta \geq 1$ be as in Lemma 2.3. Then under Assumption 1.1 (1) (2) (3) we have

$$\inf_{z \in B(x, a_k)} P_z^\omega \left(\sup_{0 \leq s \leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k \right) \geq \rho_1^{\lambda_k} \quad (2.16)$$

for almost all $\omega \in \Omega$, all k with $\max_{z \in B(x, 4\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$.

Proof. We can see from Lemma 2.3 that

$$P_z^\omega \left(\sup_{0 \leq s \leq u_k} d(z, Y_s^\omega) \leq 3\eta a_k \right) \geq c_1 \exp \left[-c_2 \frac{u_k}{a_k^\beta} \right] \geq \rho_1^{\lambda_k}$$

for all $k \geq 1$ with $\max_{v \in B(z, 3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$. Hence (2.16) holds for k with $\max_{z \in B(x, a_k)} \max_{v \in B(z, 3\eta a_k)} N_v(\omega) \leq a_k^{1/\beta}$. \square

Lemma 2.5. Suppose Assumption 1.1 (1) (3). Then there exist positive constants c_1, c_2 such that

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \leq r \right) \leq c_1 \exp \left(-c_2 \frac{t}{r^\beta} \right)$$

for almost all environment $\omega \in \Omega$, all $x \in V(G^\omega)$, $t \geq 1$ and $r \geq 1$ with $\max_{y \in B(x, r)} N_y(\omega) \leq 2r$.

Proof. The proof is quite similar to that of [22, Lemma 3.2], so we omit it. \square

We will need the following version of 0-1 law.

Theorem 2.6 (0-1 law for tail events). *For almost all environment $\omega \in \Omega$, the following holds; Let A^ω be a tail event, i.e. $A^\omega \in \bigcap_{t=0}^{\infty} \sigma\{Y_s^\omega : s \geq t\}$. Then either $P_x^\omega(A^\omega) = 0$ for all x or $P_x^\omega(A^\omega) = 1$ for all x .*

The proof of the above theorem is quite similar to that of [7, Proposition 2.3] (see also [3, Theorem 4]), so we omit the proof here.

2.2 Green function

In this subsection, we deduce Green function estimates. We define Green function as

$$g^\omega(x, y) = \int_0^\infty q_t^\omega(x, y) dt. \quad (2.17)$$

Recall that $\theta^\omega(x) = \pi^\omega(x)$ in the case of CSRW and $\theta^\omega(x) = 1$ in the case of VSRW.

Proposition 2.7. *Let $\alpha > \beta$ and suppose Assumption 1.1 (1) (2) (4). In addition we assume there exists a positive constant $c > 0$ such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$ in the case of CSRW. Then there exist positive constants c_1, c_2 such that*

$$\frac{c_1}{d(x, y)^{\alpha-\beta}} \leq g^\omega(x, y) \leq \frac{c_2}{d(x, y)^{\alpha-\beta}} \quad (2.18)$$

for almost all $\omega \in \Omega$, all $x, y \in V(G^\omega)$ with $d(x, y) \geq N_x(\omega) \wedge N_y(\omega)$.

Proof. This proof is similar to [6, Proposition 6.2]. We first prove the upper bound of (2.18).

$$\begin{aligned} g^\omega(x, y) &= \int_0^{(c_{4.3}d(x, y)) \wedge N_x(\omega)} q_t^\omega(x, y) dt + \int_{(c_{4.3}d(x, y)) \wedge N_x(\omega)}^{N_x(\omega)} q_t^\omega(x, y) dt + \int_{N_x(\omega)}^{d(x, y)} q_t^\omega(x, y) dt + \int_{d(x, y)}^\infty q_t^\omega(x, y) dt \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned} \quad (2.19)$$

We estimate J_1, J_2, J_3, J_4 as follows.

$$\begin{aligned} J_1 &\leq \int_0^{(c_{4.3}d(x, y)) \wedge N_x(\omega)} \frac{c_{4.4}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp[-c_{4.5}d(x, y)] dt \quad (\text{use (1.7)}) \\ &\leq c_1 d(x, y) \exp[-c_2 d(x, y)], \\ J_2 &\leq \int_{(c_{4.3}d(x, y)) \wedge N_x(\omega)}^{N_x(\omega)} \frac{c_{4.1}}{\sqrt{\theta^\omega(x)\theta^\omega(y)}} \exp\left[-c_{4.2} \frac{d(x, y)^2}{t}\right] dt \quad (\text{use (1.7)}) \\ &\leq c_3 N_x(\omega) \exp\left[-c_4 \frac{d(x, y)^2}{N_x(\omega)}\right] \leq c_3 d(x, y) \exp[-c_4 d(x, y)] \quad (\text{use } d(x, y) \geq N_x(\omega)), \\ J_3 &\leq \int_{N_x(\omega)}^{d(x, y)} c_{1.3} \exp[-c_{1.4}d(x, y)] dt \quad (\text{use (1.4)}) \\ &\leq c_{1.3} d(x, y) \exp[-c_{1.4}d(x, y)], \\ J_4 &\leq \int_{d(x, y)}^\infty \frac{c_{1.1}}{t^{\alpha/\beta}} \exp\left[-c_{1.2} \left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt \leq \frac{c_5}{d(x, y)^{\alpha-\beta}}. \end{aligned} \quad (2.20)$$

By (2.19) and (2.20) we have $g^\omega(x, y) \leq \frac{c_6}{d(x, y)^{\alpha-\beta}}$ for $d(x, y) \geq N_x(\omega)$. Note that $g^\omega(x, y) = g^\omega(y, x)$. Thus we complete the upper bound of (2.18).

Next we prove the lower bound of (2.18). We can obtain the lower bound in the following way.

$$g^\omega(x, y) \geq \int_{d(x, y)^\beta}^\infty q_t^\omega(x, y) dt \geq \int_{d(x, y)^\beta}^\infty \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2} \left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] dt \geq \frac{c_7}{d(x, y)^{\alpha-\beta}}.$$

We thus complete the proof. \square

2.3 Consequences of Green function and Assumption 1.1

In this subsection we give some preliminary results of Assumption 1.1 (1) (2) (3) (4) in the case of $\alpha > \beta$. This subsection is based on [27, Section 4.1]. In this subsection we assume the following conditions.

Assumption 2.8. (1) $\alpha > \beta$,

(2) (CSRW case) There exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$.

Recall that Proposition 2.7 holds under Assumption 1.1 (1) (2) (4) and Assumption 2.8.

We write $e_F^\omega(x) = P_x^\omega(\sigma_F^{+\omega} = \infty) 1_F(x)$ as the equilibrium measure of $F \subset V(G^\omega)$, and define $\text{Cap}^\omega(F) = \sum_{x \in F} e_F^\omega(x) \theta^\omega(x)$ as the capacity of $F \subset V(G^\omega)$. Then we have

$$P_x^\omega(\sigma_F^{+\omega} < \infty) = \sum_{y \in F} g^\omega(x, y) e_F^\omega(y) \theta^\omega(y) \quad (2.21)$$

for any finite set F and for any $x \in V(G^\omega)$ since

$$\begin{aligned} P_x^\omega(\sigma_F^{+\omega} < \infty) &= \int_0^\infty \sum_{y \in F} P_x^\omega(Y_t^\omega = y, Y_s^\omega \notin F \text{ for any } s > t) dt \quad (\text{last exit decomposition}) \\ &= \int_0^\infty \sum_{y \in F} q_t^\omega(x, y) \theta^\omega(y) P_y^\omega(\sigma_F^{+\omega} = \infty) dt \quad (\text{by the Markov property}) \\ &= \sum_{y \in F} g^\omega(x, y) e_F^\omega(y) \theta^\omega(y). \end{aligned}$$

Lemma 2.9. Under Assumption 1.1 (1) (2) (3) (4) and Assumption 2.8, there exists a positive constant c such that

$$\text{Cap}^\omega(B^\omega(x, 2r)) \geq cr^{\alpha-\beta}$$

for almost all $\omega \in \Omega$, all $x \in V(G^\omega)$ and $r \geq 1$ with $r \geq \max_{v \in B(x, r)} N_v(\omega)$.

Proof. Recall the notations in (1.3).

$$\begin{aligned} 1 &= \frac{1}{\theta^\omega(B(x, r))} \sum_{y \in B^\omega(x, r)} P_y^\omega(\sigma_{B(x, 2r)}^{+\omega} < \infty) \theta^\omega(y) \\ &= \frac{1}{\theta^\omega(B(x, r))} \sum_{y \in B^\omega(x, r)} \sum_{\substack{z \in B^\omega(x, 2r) \\ d(x, z) = 2r}} g^\omega(y, z) e_{B^\omega(x, 2r)}^\omega(z) \theta^\omega(z) \theta^\omega(y) \quad (\text{we use (2.21)}) \\ &\leq \frac{c_1}{\theta^\omega(B(x, r))} \frac{1}{r^{\alpha-\beta}} \sum_{\substack{z \in B^\omega(x, 2r) \\ d(x, z) = 2r}} \sum_{y \in B^\omega(x, r)} e_{B^\omega(x, 2r)}^\omega(z) \theta^\omega(z) \theta^\omega(y) \\ &\quad (\text{since } d(y, z) \geq r \geq N_y(\omega) \text{ and Proposition 2.7}) \\ &= \frac{c_1}{\theta^\omega(B(x, r))} \frac{\theta^\omega(B(x, r))}{r^{\alpha-\beta}} \sum_{\substack{z \in B^\omega(x, 2r) \\ d(x, z) = 2r}} e_{B^\omega(x, 2r)}^\omega(z) \theta^\omega(z) \\ &= \frac{c_1}{r^{\alpha-\beta}} \text{Cap}^\omega(B^\omega(x, 2r)). \end{aligned}$$

We thus complete the proof. \square

Recall the notations in (1.3) and set

$$\begin{aligned} \gamma_{x, F}^\omega(K_1) &= P_x^\omega(Y_{\sigma_F^+}^\omega \in K_1), \\ \pi_{x, F}^\omega(dt, K_2) &= P_x^\omega(Y_{\sigma_F^+}^\omega \in K_2, \sigma_F \in dt) \end{aligned}$$

for $F, K_1, K_2 \subset V(G^\omega)$. Note that $\int_0^\infty \pi_{x, F}^\omega(dt, K) = \gamma_{x, F}^\omega(K)$ and $\gamma_{x, F}^\omega(F) = P_x^\omega(\sigma_F^{+\omega} < \infty)$.

Lemma 2.10. For almost all $\omega \in \Omega$,

$$g^\omega(x, y) = \sum_{v \in F^\omega} g^\omega(v, y) \gamma_{x, F^\omega}^\omega(v) \quad (2.22)$$

for any finite set $F^\omega \subset V(G^\omega)$, $x \notin F^\omega$ and $y \in F^\omega$. In particular we have

$$P_x^\omega(Y_t^\omega \in F^\omega \text{ for some } t > 0) \leq \inf_{y \in F^\omega} \left(\frac{g^\omega(x, y)}{\inf_{z \in F^\omega} g^\omega(z, y)} \right). \quad (2.23)$$

Proof. We write $F = F^\omega$ and $\sigma = \sigma_{F^\omega}^+ = \inf\{t > 0 \mid Y_t^\omega \in F\}$ for notational simplification. Then for any $x \notin F$, $y \in F$ we have

$$\begin{aligned} P_x^\omega(Y_t^\omega = y) &= E_x^\omega \left[1_{\{\sigma \leq t\}} P_{Y_\sigma^\omega}^\omega(Y_{t-\sigma}^\omega = y) \right] = \sum_{v \in F} E_x^\omega \left[1_{\{\sigma \leq t\}} 1_{\{Y_\sigma^\omega = v\}} P_{Y_\sigma^\omega}^\omega(Y_{t-\sigma}^\omega = y) \right] \\ &= \sum_{v \in F} \int_0^t P_v^\omega[Y_{t-s}^\omega = y] \pi_{x, F}^\omega(ds, v). \end{aligned}$$

Hence we have

$$\begin{aligned} g^\omega(x, y) &= \int_0^\infty \sum_{v \in F} \int_0^t q_{t-s}^\omega(v, y) \pi_{x, F}^\omega(ds, v) dt = \int_0^\infty \sum_{v \in F} \int_s^\infty q_{t-s}^\omega(v, y) dt \pi_{x, F}^\omega(ds, v) \\ &= \int_0^\infty \sum_{v \in F} g^\omega(v, y) \pi_{x, F}^\omega(ds, v) = \sum_{v \in F} g^\omega(v, y) \gamma_{x, F}^\omega(v). \end{aligned}$$

We thus complete the proof of (2.22). (2.23) is immediate from (2.22). \square

Lemma 2.11. Under Assumption 1.1 (1) (2) (3) (4) and Assumption 2.8 there exist positive constants c_1, c_2 such that for almost all $\omega \in \Omega$ the following hold.

- (1) $P_x^\omega(\sigma_{B(x_0, 2r)}^{+\omega} < \infty) \leq c_1 \frac{r^{\alpha-\beta}}{(d(x, x_0) - r)^{\alpha-\beta}}$ for all $x, x_0 \in V(G^\omega)$, $r \geq 1$ with $d(x, x_0) \geq 2r + 1$ and $r \geq \max_{v \in B(x_0, r)} N_v(\omega)$.
- (2) $P_x^\omega(\sigma_{B(x_0, 2r)}^{+\omega} < \infty) \geq c_2 \frac{r^{\alpha-\beta}}{(d(x, x_0) + 2r)^{\alpha-\beta}}$ for all $x, x_0 \in V(G^\omega)$, $r \geq 1$ with $d(x, x_0) \geq 2r$, $r \geq N_x(\omega)$ and $r \geq \max_{v \in B(x_0, r)} N_v(\omega)$.

Proof. We first prove (1) by using (2.23). Let $x, x_0 \in V(G^\omega)$ satisfy $d(x, x_0) \geq 2r + 1$. For any $y \in B(x_0, r)$ we have

$$d(x, y) \geq d(x, x_0) - d(x_0, y) \geq d(x, x_0) - r \geq 2r - r = r.$$

By Proposition 2.7, for any $y \in B(x_0, r)$ and for any r with $r \geq \max_{y \in B(x_0, r)} N_y(\omega)$ we have

$$g^\omega(x, y) \leq \frac{c_1}{d(x, y)^{\alpha-\beta}} \leq \frac{c_1}{(d(x, x_0) - r)^{\alpha-\beta}}. \quad (2.24)$$

Next note that $B(x_0, 2r) \subset B(y, 3r)$ for any $y \in B(x_0, r)$. Since $g^\omega(\cdot, y)$ is superharmonic function, using minimal principle and Proposition 2.7 we have

$$\inf_{z \in B^\omega(x_0, 2r)} g^\omega(z, y) \geq \inf_{z \in B^\omega(y, 3r)} g^\omega(z, y) \geq \inf_{\substack{z \in B^\omega(y, 3r+1) \\ d(y, z) = 3r+1}} g^\omega(z, y) \geq \frac{c_2}{r^{\alpha-\beta}} \quad (2.25)$$

for all $r \geq 1$ and $y \in B(x_0, r)$ with $3r + 1 \geq \max_{v \in B(x_0, r)} N_v(\omega)$. Hence by (2.23), (2.24) and (2.25) we have

$$P_x^\omega(\sigma_{B(x_0, 2r)}^{+\omega} < \infty) \leq \inf_{y \in B(x_0, r)} \left(\frac{g^\omega(x, y)}{\inf_{z \in B(x_0, 2r)} g^\omega(z, y)} \right) \leq c_3 \frac{r^{\alpha-\beta}}{(d(x, x_0) - r)^{\alpha-\beta}}$$

for all r with $r \geq \max_{v \in B(x_0, r)} N_v(\omega)$. Thus we complete the proof of (1).

Next we prove (2). Note that

$$\begin{aligned} P_x^\omega \left(\sigma_{B(x_0, 2r)}^{+\omega} < \infty \right) &= \sum_{y \in B^\omega(x_0, 2r)} g^\omega(x, y) e_{B^\omega(x_0, 2r)}^\omega(y) \theta^\omega(y) \quad (\text{use (2.21)}) \\ &\geq \left(\inf_{y \in B^\omega(x_0, 2r)} g^\omega(x, y) \right) \sum_{y \in B^\omega(x_0, 2r)} e_{B^\omega(x_0, 2r)}^\omega(y) \theta^\omega(y) = \left(\inf_{y \in B^\omega(x_0, 2r)} g^\omega(x, y) \right) \text{Cap}^\omega(B(x_0, 2r)). \end{aligned}$$

By $B(x_0, 2r) \subset B(x, d(x, x_0) + 2r)$, the minimum principle for superharmonic functions and our assumptions we have

$$\inf_{y \in B^\omega(x_0, 2r)} g^\omega(x, y) \geq \inf_{y \in B^\omega(x, d(x, x_0) + 2r)} g^\omega(x, y) \geq \inf_{\substack{y \in B^\omega(x, d(x, x_0) + 2r + 1) \\ d(y, x) = d(x, x_0) + 2r + 1}} g^\omega(x, y) \geq \frac{c_4}{(d(x, x_0) + 2r)^{\alpha - \beta}}$$

for $r \geq N_x(\omega)$. By Lemma 2.9 $\text{Cap}^\omega(B(x_0, r)) \geq c_5 r^{\alpha - \beta}$ for $r \geq \max_{v \in B(x_0, r)} N_v(\omega)$. Hence

$$P_x^\omega \left(\sigma_{B(x_0, 2r)}^{+\omega} < \infty \right) \geq \frac{c_6 r^{\alpha - \beta}}{(d(x, x_0) + 2r)^{\alpha - \beta}}$$

for $r \geq N_x(\omega)$ and $r \geq \max_{v \in B(x_0, r)} N_v(\omega)$. We thus complete the proof. \square

Lemma 2.12. *Under Assumption 1.1 (1) (2) (3) (4) and Assumption 2.8 there exist positive constants c_1 and T_0 such that*

$$P_x^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) \leq \frac{c_1 r^{\alpha - \beta} t}{t^{\alpha / \beta}}$$

for almost all $\omega \in \Omega$, all $t \geq T_0$, $r \geq 1$ and $x, x_0 \in V(G^\omega)$ with $t^{1/\beta} \geq r$, $d(x, x_0) \leq r$ and $r \geq \max_{z \in B(x_0, r)} N_z(\omega)$.

Proof. First note that

$$\begin{aligned} P_x^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) &= \sum_{y \in V(G^\omega)} P_x^\omega (Y_t^\omega = y) P_y^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &= \sum_{y; t^{1/\beta} < d(x_0, y) - r} P_x^\omega (Y_t^\omega = y) P_y^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &\quad + \sum_{y; r < d(x_0, y) - r \leq t^{1/\beta}} P_x^\omega (Y_t^\omega = y) P_y^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &\quad + \sum_{y; d(x_0, y) \leq 2r} P_x^\omega (Y_t^\omega = y) P_y^\omega (d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &:= J_1 + J_2 + J_3. \end{aligned}$$

We estimate J_1, J_2 and J_3 in the following way.

For $t, r \geq 1$ with $t \geq N_x(\omega)$ and $r \geq \max_{z \in B(x_0, r)} N_z$ (note that $t \geq N_x(\omega)$ follows from our assumptions),

using (1.4), Lemma 2.11, (1.6) we have

$$\begin{aligned}
J_1 &\leq \sum_{y; t^{1/\beta} < d(x_0, y) - r} \frac{c_1 r^{\alpha-\beta}}{(d(y, x_0) - r)^{\alpha-\beta}} \\
&\quad \cdot \left\{ \frac{c_{1.1}}{t^{\alpha/\beta}} \exp \left[-c_{1.2} \left(\frac{d(x, y)}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right] + c_{1.3} \exp [-c_{1.4} d(x, y)] \right\} \theta^\omega(y) \quad (\text{use (1.4) and Lemma 2.11}) \\
&\leq \sum_{\ell=1}^{\infty} \sum_{y; d(x_0, y) \in [\ell t^{1/\beta} + r, (\ell+1)t^{1/\beta} + r]} \frac{c_2 r^{\alpha-\beta}}{(d(y, x_0) - r)^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}} \exp \left[-c_{1.2} \left(\frac{d(y, x_0) - r}{t^{1/\beta}} \right)^{\frac{\beta}{\beta-1}} \right] \theta^\omega(y) \\
&\quad + \sum_{\ell=1}^{\infty} \sum_{y; d(x_0, y) \in [\ell t^{1/\beta} + r, (\ell+1)t^{1/\beta} + r]} \frac{c_3 r^{\alpha-\beta}}{(d(y, x_0) - r)^{\alpha-\beta}} \exp [-c_{1.4} (d(y, x_0) - r)] \theta^\omega(y) \\
&\quad (\text{since } d(x, y) \geq d(y, x_0) - d(x_0, x) \text{ and } d(x_0, x) \leq r) \\
&\leq \sum_{\ell=1}^{\infty} \frac{c_2 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \frac{1}{t^{\alpha/\beta}} \exp [-c_{1.2} \ell^{\frac{\beta}{\beta-1}}] \theta^\omega(B(x_0, (\ell+1)t^{1/\beta} + r)) \\
&\quad + \sum_{\ell=1}^{\infty} \frac{c_3 r^{\alpha-\beta}}{(\ell t^{1/\beta})^{\alpha-\beta}} \exp [-c_{1.4} \ell t^{1/\beta}] \theta^\omega(B(x_0, (\ell+1)t^{1/\beta} + r)) \\
&\leq \frac{c_4 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} \sum_{\ell=1}^{\infty} \ell^\beta \exp [-c_{1.2} \ell^{\beta/\beta-1}] + \frac{c_5 r^{\alpha-\beta}}{t^{\alpha/\beta-1}} t^{\alpha/\beta} \sum_{\ell=1}^{\infty} \ell^\beta \exp [-c_{1.4} \ell t^{1/\beta}] \\
&\quad (\text{use } \theta^\omega(B(x_0, (\ell+1)t^{1/\beta} + r)) \leq c(\ell t^{1/\beta})^\alpha \text{ since } t^{1/\beta} \geq r) \\
&\leq \frac{c_6 r^{\alpha-\beta}}{t^{\alpha/\beta-1}}, \quad (\text{since } t \mapsto t^{\alpha/\beta} \sum_{\ell=1}^{\infty} \ell^\beta \exp [-c_{1.4} \ell t^{1/\beta}] \text{ is bounded}).
\end{aligned}$$

Next we see J_2 . First, set $\phi_r(k) = (r+k)^\beta$ ($k, r \geq 1$). We can easily see that there exist a positive constant $K_1 = K_1(r) > 1$ such that

$$\phi_r(k) \leq \frac{1}{2} \phi_r(K_1 k) \quad (2.26)$$

for all $k \geq 1$. Using this inequality we see that for $r \geq N_{x_0}(\omega)$

$$\begin{aligned}
&\sum_{y; r \leq d(x_0, y) - r \leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y, x_0) - r)^{\alpha-\beta}} \leq \sum_{k=2r}^{r+t^{1/\beta}} \frac{\theta^\omega(B(x_0, k) \setminus B(x_0, k-1))}{(k-r)^{\alpha-\beta}} \\
&\leq c_7 + \sum_{\ell=1}^{(t^{1/\beta}-r)/K_1} \sum_{k \in [2r+\ell K_1, 2r+(\ell+1)K_1]} \frac{\theta^\omega(B(x_0, k) \setminus B(x_0, k-1))}{(k-r)^{\alpha-\beta}} \leq c_7 + c_8 \sum_{\ell=1}^{(t^{1/\beta}-r)/K_1} (r+\ell K_1)^\beta \\
&\leq c_7 + c_9 t, \quad (\text{use (2.26)}). \quad (2.27)
\end{aligned}$$

We go back to estimate J_2 . Note that for y with $r \leq d(x_0, y) - r \leq t^{1/\beta}$ we see $d(x, y) \leq d(x, x_0) + d(x_0, y) \leq 3t^{1/\beta}$. For $r \geq 1, t \geq 1$ with $t \geq T_0 := 3^{\beta/(\beta-1)}$ (so that $3t^{1/\beta} \leq t$ for $t \geq T_0$) and $r \geq \max_{z \in B(x_0, r)} N_z(\omega)$ (in particular $t \geq N_x(\omega)$), using Lemma 2.11, (1.4) and (2.27) we have

$$\begin{aligned}
J_2 &\leq \sum_{y; r < d(x_0, y) - r \leq t^{1/\beta}} \frac{c_{10} r^{\alpha-\beta}}{(d(y, x_0) - r)^{\alpha-\beta}} \frac{\theta^\omega(y)}{t^{\alpha/\beta}} = \frac{c_{10} r^{\alpha-\beta}}{t^{\alpha/\beta}} \sum_{y; r \leq d(x_0, y) - r \leq t^{1/\beta}} \frac{\theta^\omega(y)}{(d(y, x_0) - r)^{\alpha-\beta}} \\
&\leq \frac{c_{11} r^{\alpha-\beta} t}{t^{\alpha/\beta}}, \quad (\text{use (2.27)}).
\end{aligned}$$

Finally we see J_3 . For $t \geq T_0 := 3^{\beta/(\beta-1)}$, $N_x(\omega) \leq t$ and $N_x(\omega) \leq r$, using (1.4) we have

$$\begin{aligned} J_3 &\leq \sum_{y: d(y, x_0) \leq 2r} P_x^\omega(Y_t^\omega = y) = \sum_{y: d(y, x_0) \leq 2r} q_t^\omega(x, y) \theta^\omega(y) \\ &\leq \sum_{y: d(x, y) \leq 3r} q_t^\omega(x, y) \theta^\omega(y) \leq \frac{c_{12} r^\alpha}{t^{\alpha/\beta}} \leq \frac{c_{12} r^{\alpha-\beta} t}{t^{\alpha/\beta}}. \end{aligned}$$

We thus complete the proof. \square

Lemma 2.13. *Under Assumption 1.1 (1) (2) (3) (4) and Assumption 2.8 there exist constants $c_1 > 0, c_2, T_0 \geq 1$ such that*

$$P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) \geq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, all $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ with $d(x, x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. Take a constant c_2 such that $c_{3.1} c_2^\alpha - c_{3.2} 2^\alpha > 0$. Note that by (1.6) we have $\theta^\omega(\{y \in V(G) \mid d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]\}) \geq (c_{3.1} c_2^\alpha - c_{3.2} 2^\alpha) t^{\alpha/\beta}$, and for y and sufficiently large t (say $t \geq T_0$) with $d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]$ we have $d(x, y)^{1+\epsilon} \leq (d(x, x_0) + d(x_0, y))^{1+\epsilon} \leq \{(c_2 + 1)t^{1/\beta}\}^{1+\epsilon} \leq t$ since $1 + \epsilon < \beta$ (see Assumption 1.1). Then by Lemma 2.11 (2), (1.5), (1.6), for t, r as in the statement above we have

$$\begin{aligned} P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) &= \sum_{y \in V(G^\omega)} q_t^\omega(x, y) \theta^\omega(y) P_y^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &\geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} q_t^\omega(x, y) \theta^\omega(y) P_y^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > 0) \\ &\geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_{2.1}}{t^{\alpha/\beta}} \exp\left[-c_{2.2} \left(\frac{d(x, y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right] \theta^\omega(y) \frac{c_3 r^{\alpha-\beta}}{(d(x_0, y) + 2r)^{\alpha-\beta}} \\ &\quad (\text{use (1.5), Lemma 2.11 and } d(x, y)^{1+\epsilon} \leq t, \text{ note that } t \geq N_x(\omega) \text{ follows from our assumptions}) \\ &\geq \sum_{y: d(x_0, y) \in [2t^{1/\beta}, c_2 t^{1/\beta}]} \frac{c_4}{t^{\alpha/\beta}} \theta^\omega(y) \frac{r^{\alpha-\beta}}{(t^{1/\beta})^{\alpha-\beta}} \\ &\quad \left(\text{use } d(x, y) \leq d(x, x_0) + d(x_0, y) \leq (c_2 + 1)t^{1/\beta} \text{ for } y \in B(x_0, c_2 t^{1/\beta})\right) \\ &\geq \frac{c_5(c_{3.1} c_1^\alpha - c_{3.2} 2^\alpha) r^{\alpha-\beta} t}{t^{\alpha/\beta}}. \end{aligned}$$

We thus complete the proof by taking $c_1 = c_5(c_{3.1} c_2^\alpha - c_{3.2} 2^\alpha)$. \square

Lemma 2.14. *Under Assumption 1.1 (1) (2) (3) (4) and Assumption 2.8 there exist positive constants c_1, c_2, η_0, T_0 such that for any $\eta \geq \eta_0$ the following holds;*

$$P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s \in (t, \eta t]) \geq \frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for almost all $\omega \in \Omega$, all $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ with $d(x, x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_2 t^{1/\beta})} N_z(\omega)$.

Proof. By Lemma 2.12 and Lemma 2.13 there exist positive constants c_1, c_2, c_3, T_0 such that for almost all $\omega \in \Omega$

$$\frac{c_1 r^{\alpha-\beta} t}{t^{\alpha/\beta}} \leq P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) \leq \frac{c_2 r^{\alpha-\beta} t}{t^{\alpha/\beta}}$$

for $r \geq 1$, $t \geq T_0$, $x, x_0 \in V(G^\omega)$ with $d(x, x_0) \leq r$, $t \geq r^\beta$, $r \geq \max_{z \in B(x_0, c_3 t^{1/\beta})} N_z(\omega)$. Take η_0 such that $c_2 - \frac{c_1}{\eta^{\alpha/\beta-1}} > \frac{c_2}{2}$ for all $\eta \geq \eta_0$. Then we have

$$\begin{aligned} & P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s \in (t, \eta t]) \\ & \geq P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > t) - P_x^\omega(d(x_0, Y_s^\omega) \leq 2r \text{ for some } s > \eta t) \\ & \geq c_2 \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} - c_1 \frac{r^{\alpha-\beta}(\eta t)}{(\eta t)^{\alpha/\beta}} = \frac{r^{\alpha-\beta} t}{t^{\alpha/\beta}} \left(c_2 - \frac{c_1}{\eta^{\alpha/\beta-1}} \right). \end{aligned}$$

We complete the proof by adjusting the constants. \square

2.4 Consequences of Assumption 1.3

In this subsection, we give easy consequences of Assumption 1.3. We use $\varphi(q) = \varphi_C(q) = Cq^{1/\beta}(\log \log q)^{1-1/\beta}$ in this subsection.

Lemma 2.15. (1) Under Assumption 1.3 (1), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(1)}(\omega) = L_{x, \epsilon, \gamma_1, \gamma_2, q}^{(1)}(\omega)$ such that

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega), \quad \gamma_1 \varphi(q^n) \geq \max_{y \in B(x, \gamma_2 \varphi(q^n))} N_y(\omega),$$

for all $n \geq L^{(1)}(\omega)$.

(2) Under Assumption 1.3 (2), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(2)}(\omega) = L_{x, \epsilon, \gamma_1, \gamma_2, q}^{(2)}(\omega)$ such that

$$\gamma_1 q^{n/\beta} \geq \max_{y \in B(x, \gamma_2 q^n)} N_y(\omega)$$

for all $n \geq L^{(2)}(\omega)$.

(3) Set $\psi(t) := t^{1/\beta} h(t)$, where $h(t)$ is non-increasing and $\psi(t)$ is increasing function. Under Assumption 1.3 (3), for all $\gamma_1, \gamma_2 > 0$, $q > 1$ and for almost all $\omega \in \Omega$ there exists a positive number $L^{(3)}(\omega) = L_{x, \epsilon, \gamma_1, \gamma_2, q}^{(3)}(\omega)$ such that

$$\gamma_1 \psi(q^n) \geq \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y(\omega)$$

for all $n \geq L^{(3)}(\omega)$.

Proof. We can prove (1) (2) (3) similarly, so we prove only the first inequality in (1). Since

$$\begin{aligned} \mathbb{P} \left(\gamma_1 q^{n/\beta} < \max_{y \in B(x, \gamma_2 q^{n/\beta})} N_y \right) & \leq \sum_{y \in B(x, \gamma_2 q^{n/\beta})} \mathbb{P} \left(\gamma_1 q^{n/\beta} < N_y \right) \\ & \leq c(\gamma_2 q^{n/\beta})^\alpha f(\gamma_1 q^{n/\beta}), \end{aligned}$$

where we use union bound in the first inequality and use (1.2) in the second inequality. The conclusion follows by the Borel-Cantelli lemma. \square

3 Proof of Theorem 1.4

In this section we give the proof of Theorem 1.4.

3.1 Proof of the LIL

We follow the strategy as in [15].

Theorem 3.1. *Let $\varphi(t) = \varphi_C(t) = Ct^{1/\beta}(\log \log t)^{1-1/\beta}$, where $C > 2^{1+1/\beta}c_{1.2}^{-(\beta-1)/\beta}$. Then under Assumption 1.1 (1) (2) (3) and Assumption 1.3 (1) the following hold for almost all $\omega \in \Omega$;*

$$\limsup_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{\varphi(t)} \leq 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \quad (3.1)$$

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(x, Y_s^\omega) \leq \varphi(t) \text{ for sufficiently large } t \right) = 1, \quad \text{for all } x \in V(G^\omega). \quad (3.2)$$

In particular, we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} &\leq 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega), \\ P_x^\omega (d(x, Y_t^\omega) \leq \varphi(t) \text{ for all sufficient large } t) &= 1, \quad \text{for all } x \in V(G^\omega). \end{aligned}$$

Proof. Take $\eta > 0$ and $\delta \in (0, c_{1.2} \wedge c_{1.4})$ sufficiently small constants which satisfy

$$C > 2^{1/\beta}(1+\eta)^{1/\beta} \left(\frac{1}{c_{1.2} - \delta} \right)^{(\beta-1)/\beta}. \text{ Set } t_n = (1+\eta)^n.$$

First we estimate $P_x^\omega \left(\sup_{0 \leq s \leq t_{n+1}} d(x, Y_s^\omega) \geq 2\varphi_C(t_n) \right)$. For all $\delta \in (0, c_{1.2} \wedge c_{1.4})$, using Lemma 2.2 we have

$$\begin{aligned} P_x^\omega \left(\sup_{0 \leq s \leq t_{n+1}} d(x, Y_s^\omega) \geq 2\varphi(t_n) \right) &\leq c_1 \exp \left[-(c_{1.2} - \delta) \left(\frac{\varphi(t_n)}{(2t_{n+1})^{1/\beta}} \right)^{\beta/(\beta-1)} \right] + c_2 \exp [-c_3 t_{n+1}] \\ &\leq c_1 \exp \left[-(c_{1.2} - \delta) \left(\frac{\varphi(t_n)}{(2(1+\eta)t_n)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] + c_2 \exp [-c_3 t_{n+1}] \end{aligned} \quad (3.3)$$

for $\sup_{z \in B(x, 2\varphi(t_n))} N_z(\omega) \leq \varphi(t_n) \wedge t_{n+1}$. Note that $\sup_{z \in B(x, 2\varphi(t_n))} N_z(\omega) \leq \varphi(t_n) \wedge t_{n+1}$ for all n larger than a certain constant $L = L(\omega)$ by Lemma 2.15 (1).

We prove (3.1). Let $C > 2^{1/\beta}(1+\eta)^{1/\beta} \left(\frac{1}{c_{1.2} - \delta} \right)^{(\beta-1)/\beta}$ be as above. Since the last term of (3.3) is summable by the definition of η and δ . By the Borel-Cantelli lemma we have

$$\limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq s \leq t_{n+1}} d(Y_0^\omega, Y_s^\omega)}{2\varphi(t_n)} \leq 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

For all t with $t_n \leq t < t_{n+1}$ we have

$$\frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{2\varphi(t)} \leq \frac{\sup_{0 \leq s \leq t_{n+1}} d(Y_0^\omega, Y_s^\omega)}{2\varphi(t_n)}.$$

Hence we obtain (3.1) from the above inequality and adjusting the constants.

Next we prove (3.2). Let $C > 2^{1/\beta}(1+\eta)^{1/\beta} \left(\frac{1}{c_{1.2} - \delta} \right)^{(\beta-1)/\beta}$ be as above. Since

$$P_x^\omega \left(\sup_{0 \leq s \leq t_n} d(x, Y_s^\omega) \geq 2\varphi(t_n) \right) \leq P_x^\omega \left(\sup_{0 \leq s \leq t_{n+1}} d(x, Y_s^\omega) \geq 2\varphi(t_n) \right)$$

for $t \in [t_n, t_{n+1}]$ and the last term of (3.3) is summable by the definition of η and δ . By the Borel-Cantelli lemma we have

$$P_x^\omega \left(\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega) \leq 2\varphi(t) \text{ for all sufficiently large } t \right) = 1, \quad \text{for all } x \in V(G^\omega). \quad (3.4)$$

We thus complete the (3.2) by adjusting the constants. \square

Theorem 3.2. Let $\varphi(t) = \varphi_C(t) = Ct^{1/\beta}(\log \log t)^{1-1/\beta}$, where $0 < C < \frac{1}{2^{1+1/\beta}} \left(\frac{c_{3.1}}{c_{3.2}} \right)^{1/\alpha} \left(\frac{1}{c_{2.2}} \right)^{(\beta-1)/\beta}$. Then under Assumption 1.1 (1) (2) (3) and Assumption 1.3 (1) the following holds;

$$\limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \geq 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega).$$

In particular, we have

$$\begin{aligned} P_x^\omega(d(Y_0^\omega, Y_t^\omega) \geq \varphi(t) \text{ for sufficiently large } t) &= 1, \quad \text{for all } x \in V(G^\omega), \\ \limsup_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{\varphi(t)} &\geq 1, \quad P_x^\omega\text{-a.s. for all } x \in V(G^\omega). \end{aligned}$$

Proof. Define $\Phi(q) = q^{1/\beta}(\log \log q)^{1-1/\beta}$ and let C be as above. Take $\eta > 0$ as a sufficiently small constant such that

$$C < \frac{1}{2^{1/\beta}} \left\{ \frac{1}{2} \left(\frac{c_{3.1}}{c_{3.2}} \right)^{1/\alpha} - \eta \right\} \left(\frac{1}{c_{2.2}} \right)^{(\beta-1)/\beta}.$$

Set $\frac{1}{\lambda} = \frac{1}{2} \left(\frac{c_{3.1}}{c_{3.2}} \right)^{1/\alpha} - \eta$. Note that $c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha > 0$ and $c_{2.2}(2^{1/\beta}C\lambda)^{\beta/(\beta-1)} < 1$.

We prove that

$$\sum_n P_x^\omega(A_n^\omega \mid \mathcal{F}_{2^n}^\omega) = \infty, \quad (3.5)$$

where $A_n^\omega = \{d(Y_{2^n}^\omega, Y_{2^{n+1}}^\omega) \geq 2\varphi(2^{n+1})\}$ and $\mathcal{F}_t^\omega = \sigma(Y_s^\omega \mid s \leq t)$. To prove (3.5), first note that by Theorem 3.1 there exists a sufficiently large constant C_1 such that for almost all $\omega \in \Omega$

$$d(x, Y_{2^n}^\omega) \leq C_1\Phi(2^n) \quad \text{for sufficiently large } n \text{ (say } n \geq \tilde{N}_1), \quad P_x^\omega\text{-a.s.}$$

Set $B_n^\omega = A_n^\omega \cap \{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}$. Then we have

$$\begin{aligned} P_x^\omega(A_n^\omega \mid \mathcal{F}_{2^n}^\omega) &\geq P_x^\omega(B_n^\omega \mid \mathcal{F}_{2^n}^\omega) = 1_{\{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}} P_{Y_{2^n}^\omega}^\omega(d(Y_0^\omega, Y_{2^{n+1}-2^n}^\omega) \geq 2\varphi(2^{n+1})) \\ &\geq \left(\inf_{u \in B^\omega(x, C_1\Phi(2^n))} P_u^\omega(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1})) \right) \cdot 1_{\{d(Y_0^\omega, Y_{2^n}^\omega) \leq C_1\Phi(2^n)\}}, \quad P_x^\omega\text{-a.s.} \end{aligned} \quad (3.6)$$

We consider the first term of (3.6). Take $u \in B^\omega(x, C_1\Phi(2^n))$. Since $1 + \epsilon < \beta$, there exists a positive integer $\tilde{N}_2 = \tilde{N}_2(\lambda)$ (which does not depend on u, ω) such that $d(u, v)^{1+\epsilon} \leq 2^n$ for all $n \geq \tilde{N}_2$ and $v \in B^\omega(u, \lambda\varphi(2^{n+1}))$. So for all $n \geq \tilde{N}_2$ with $2^n \wedge 2\varphi(2^{n+1}) \geq N_u(\omega)$, using (1.5) and (1.6) we have

$$\begin{aligned} P_u^\omega(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1})) &\geq P_u^\omega(2\varphi(2^n) \leq d(Y_0^\omega, Y_{2^n}^\omega) \leq \lambda\varphi(2^{n+1})) \\ &= \sum_{\substack{v \in V(G^\omega) \\ \varphi(2^{n+1}) \leq d(u, v) \leq \lambda\varphi(2^{n+1})}} q_{2^n}^\omega(u, v) \theta^\omega(v) \\ &\geq \sum_{\substack{v \in V(G^\omega) \\ 2\varphi(2^{n+1}) \leq d(u, v) \leq \lambda\varphi(2^{n+1})}} \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp \left[-c_{2.2} \left(\frac{d(u, v)}{(2^n)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \theta^\omega(v) \\ &\geq \frac{c_{2.1}}{(2^n)^{\alpha/\beta}} \exp \left[-c_{2.2} \left(\frac{\lambda\varphi(2^{n+1})}{(2^n)^{1/\beta}} \right)^{\beta/(\beta-1)} \right] \theta^\omega(\{v \in V(G^\omega) \mid 2\varphi(2^{n+1}) \leq d(u, v) \leq \lambda\varphi(2^{n+1})\}) \\ &\geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha \left(\frac{1}{(n+1)\log 2} \right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/\beta-1}} (\log \log 2^{n+1})^{(\beta-1)\alpha/\beta}. \end{aligned}$$

By the above estimate we have

$$\begin{aligned} & \inf_{u \in B^\omega(x, C_1 \Phi(2^n))} P_u^\omega(d(Y_0^\omega, Y_{2^n}^\omega) \geq 2\varphi(2^{n+1})) \\ & \geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha \left(\frac{1}{(n+1)\log 2} \right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/\beta-1}} (\log \log 2^{n+1})^{(\beta-1)\alpha/\beta} \end{aligned} \quad (3.7)$$

for $n \geq \tilde{N}_2$ with $\max_{u \in B(x, C_1 \Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$. By Lemma 2.15 (1), $\max_{u \in B(x, C_1 \Phi(2^n))} N_u(\omega) \leq 2^n \wedge 2\varphi(2^{n+1})$ holds for sufficiently large n (say $n \geq \tilde{N}_3 = \tilde{N}_3(\omega)$). Hence by (3.6) and (3.7) we have

$$P_x^\omega(A_n^\omega | \mathcal{F}_{2^n}^\omega) \geq c_{2.1}(c_{3.1}\lambda^\alpha - c_{3.2}2^\alpha)C^\alpha \left(\frac{1}{(n+1)\log 2} \right)^{c_{2.2}(2^{1/\beta}\lambda C)^{\beta/\beta-1}} (\log \log 2^{n+1})^{(\beta-1)\alpha/\beta} \quad (3.8)$$

for $n \geq \tilde{N}_1 \vee \tilde{N}_2 \vee \tilde{N}_3$. We thus complete to show (3.5).

By (3.5) and the second Borel-Cantelli lemma, $d(x, Y_{2^n}^\omega) \geq \varphi(2^n)$ or $d(x, Y_{2^{n+1}}^\omega) \geq \varphi(2^n)$ for infinitely many n . Hence

$$\limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{\varphi(t)} \geq 1.$$

We thus complete the proof. \square

By Theorem 3.1, 3.2 and 2.6 we obtain (1.10).

3.2 Another law of the iterated logarithm

The proof of Theorem 1.4 (2) is quite similar to that of [23, Theorem 4.1] by using Lemma 2.2, Corollary 2.4, Lemma 2.5, Theorem 2.6 and Lemma 2.15 (2). So we omit the proof.

4 Lower Rate Function

In this section we give the proof of Theorem 1.5. We follow the strategy as in [27, Section 4.1].

Theorem 4.1. *Suppose that Assumption 1.1 (1) (2) (3) (4). In addition suppose that there exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$ in the case of CSRW. Let $\alpha/\beta > 1$, $h : [0, \infty) \rightarrow (0, \infty)$ be a function such that $h(t) \searrow 0$ as $t \rightarrow \infty$, $\varphi(t) := t^{1/\beta}h(t)$ be increasing for all sufficiently large t and satisfy Assumption 1.3 (3). If the function $h(t)$ satisfies*

$$\int_1^\infty \frac{1}{t} h(t)^{\alpha-\beta} dt < \infty \quad (4.1)$$

then for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$ we have

$$P_x^\omega(d(x, Y_t^\omega) \geq t^{1/\beta}h(t) \text{ for all sufficiently large } t) = 1.$$

Proof. Set $\varphi(t) := t^{1/\beta}h(t)$, $t_n := 2^n$ and $A_n^\omega := \{d(x, Y_s^\omega) \leq \varphi(s) \text{ for some } s \in (t_n, t_{n+1}]\}$. Note that there exists a constant c_1 such that $\varphi(s) \leq 2c_1\varphi(t_n)$ for all sufficiently large n (say $n \geq N_1$) and for all $s \in (t_n, t_{n+1}]$. Then by Lemma 2.12 we have

$$P_x^\omega(A_n^\omega) \leq P_x^\omega(d(x, Y_s^\omega) \leq 2c_1\varphi(t_n) \text{ for some } s > t_n) \leq \frac{c_2\varphi(t_n)^{\alpha-\beta}t_n}{t_n^{\alpha/\beta}}$$

for n with

$$\begin{aligned} n & \geq N_1, \quad 2^n \geq T_0, \text{ where } T_0 \text{ is as in Lemma 2.12, } t_n^{1/\beta} \geq c_1\varphi(t_n), \\ c_1\varphi(t_n) & \geq \max_{z \in B(x, c_1\varphi(t_n))} N_z(\omega). \end{aligned} \quad (4.2)$$

Note that (4.2) is satisfied for sufficiently large n (say $n \geq N_2 = N_2(\omega)$) by Assumption 1.3 (3) and Lemma 2.15 (3). Thus

$$\begin{aligned} \sum_{n \geq N_2(\omega)} P_x^\omega(A_n^\omega) &\leq \sum_{n \geq N_2(\omega)} \frac{c_2 \varphi(t_n)^{\alpha-\beta} t_n}{t_n^{\alpha/\beta}} = \sum_{n \geq N_2(\omega)} \frac{c_2 h(t_n)^{\alpha-\beta} t_n}{t_n} \\ &\leq \sum_{n \geq N_2(\omega)} \frac{c_3 h(t_n)^{\alpha-\beta} (t_n - t_{n-1})}{t_n} \leq c_4 \int_{t_{N_2-1}}^\infty \frac{h(s)^{\alpha-\beta}}{s} ds. \end{aligned}$$

Since the above is integrable by (4.1), by the Borel-Cantelli lemma we have

$$P_x^\omega \left(d(x, Y_t^\omega) \geq t^{1/\beta} h(t) \text{ for all sufficiently large } t \right) = 1.$$

We thus complete the proof. \square

Theorem 4.2. *Suppose that Assumption 1.1 (1) (2) (3) (4) hold. In addition suppose that there exists a positive constant c such that $\theta^\omega(x) \geq c$ for all $x \in V(G^\omega)$ in the case of CSRW. Let $\alpha/\beta > 1$, $h : [0, \infty) \rightarrow (0, \infty)$ be a function such that $h(t) \searrow 0$ as $t \rightarrow \infty$, $\varphi(t) := t^{1/\beta} h(t)$ be increasing for all sufficiently large t and satisfy Assumption 1.3 (3). If the function $h(t)$ satisfies*

$$\int_1^\infty \frac{1}{t} h(t)^{\alpha-\beta} dt = \infty \quad (4.3)$$

then for almost all $\omega \in \Omega$ and all $x \in V(G^\omega)$

$$P_x^\omega (d(x, Y_t^\omega) \geq \varphi(t) \text{ for all sufficiently large } t) = 0. \quad (4.4)$$

We cite the following form of the Borel-Cantelli Lemma (see [27, Lemma 4.15], [31, Lemma B], [11, Theorem 1]).

Lemma 4.3. *Let $\{A_k\}_{k \geq 1}$ be a family of event which satisfies the following conditions;*

- (1) $\sum_k P(A_k) = \infty$,
- (2) $P(\limsup A_k) = 0$ or 1 ,
- (3) *There exist two constants c_1, c_2 such that for each A_j there exist $A_{j_1}, \dots, A_{j_s} \in \{A_k\}_{k \geq 1}$ such that*
 - (a) $\sum_{i=1}^s P(A_j \cap A_{j_i}) \leq c_1 P(A_j)$,
 - (b) *for any $k \in \{j+1, j+2, \dots\} \setminus \{j_1, j_2, \dots, j_s\}$ we have $P(A_j \cap A_k) \leq c_2 P(A_j) P(A_k)$.*

Then infinitely many events $\{A_k\}_{k \geq 1}$ occur with probability 1.

Proof of Theorem 4.2. First we prepare preliminary facts. Since $h(t) \searrow 0$ as $t \rightarrow \infty$, there exists a positive constant T_1 such that $h(t) < 1$ for all $t \geq T_1$. So there exists a constant $\kappa \in (0, 1)$ such that $\varphi(t) \leq (\kappa t)^{1/\beta}$ for $t \geq T_1$. Take $\eta > 1 \vee \eta_0$ (where η_0 is as in Lemma 2.14) with $1 - \frac{1}{\eta} \geq \kappa$ and $c_1 = c_1(\eta) \in (0, 1)$ such that $2c_1(\eta^{n+1})^{1/\beta} \leq (\eta^n)^{1/\beta}$ for all n . Note that for all s with $\eta^{n+1} \leq s \leq \eta^{n+2}$ we have

$$\varphi(\eta^{n+1}) = (\eta^{n+1})^{1/\beta} h(\eta^{n+1}) \geq 2c_1(\eta^{n+2})^{1/\beta} h(s) \geq 2c_1 \varphi(s), \quad (4.5)$$

and for all sufficiently large i, j with $i \geq j+2$ and $\eta^j \geq T_1$ (say $j \geq N_1$) we have

$$(c_1 \varphi(\eta^{i+1}))^\beta \stackrel{(4.5)}{\leq} \varphi(\eta^i)^\beta \leq \kappa \eta^i \stackrel{1-\frac{1}{\eta} \geq \kappa}{\leq} \eta^i - \eta^{i-1} \leq \eta^i - \eta^{j+1}. \quad (4.6)$$

Now we prove (4.4). Set $A_n^\omega := \{d(Y_0^\omega, Y_s^\omega) \leq 2c_1 \varphi(\eta^{n+1}) \text{ for some } s \in (\eta^n, \eta^{n+1}]\}$. We use Lemma 4.3 to show that infinitely many A_n^ω occur with probability 1.

Note that $\eta^n \geq (c_1\varphi(\eta^{n+1}))^\beta$ for sufficiently large n (say $n \geq N_2 = N_2(\eta)$) by (4.6). By Lemma 2.14 we have

$$P_x^\omega(A_n^\omega) \geq c_2 \frac{(c_1\varphi(\eta^{n+1}))^{\alpha-\beta} \eta^n}{\eta^{n\alpha/\beta}}$$

for $\eta \geq \eta_0$ (where η_0 is as in Lemma 2.14) and $n \geq N_2$ with

$$\eta^n \geq T_0, \text{ where } T_0 \text{ is as in Lemma 2.14, } c_1\varphi(\eta^{n+1}) \geq \max_{z \in B(x, c_2\eta^{n/\beta})} N_z(\omega). \quad (4.7)$$

Note that (4.7) holds for sufficiently large n (say $n \geq N_3(\omega)$) by Assumption 1.3 (3) and Lemma 2.15 (3). Hence

$$\begin{aligned} \sum_{n \geq N_3} P_x^\omega(A_n^\omega) &\geq \sum_{n \geq N_3} \frac{c_2(c_1\varphi(\eta^{n+1}))^{\alpha-\beta} \eta^n}{\eta^{n\alpha/\beta}} = \sum_{n \geq N_3} c_2 c_1^{\alpha-\beta} \eta^{\alpha/\beta} \frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta \cdot \eta^{n+1}} \eta^{n+1} \\ &= \sum_{n \geq N_3} \frac{c_2 c_1^{\alpha-\beta} \eta^{\alpha/\beta}}{\eta \cdot (\eta-1)} \frac{h(\eta^{n+1})^{\alpha-\beta}}{\eta^{n+1}} (\eta^{n+2} - \eta^{n+1}) \geq \frac{c_2 c_1^{\alpha-\beta} \eta^{\alpha/\beta}}{\eta(\eta-1)} \int_{\eta^{N_3+1}}^\infty \frac{h(s)^{\alpha-\beta}}{s} ds. \end{aligned}$$

Thus we have $\sum_n P_x^\omega(A_n^\omega) = \infty$ by (4.5).

The condition (2) in Lemma 4.3 is immediate from Theorem 2.6, since $\limsup_k A_k^\omega$ is a tail event.

Next we show the condition (3) in Lemma 4.3. Set $\sigma_n^\omega := \inf\{t \in (\eta^n, \eta^{n+1}] \mid d(Y_0^\omega, Y_t^\omega) \leq 2c_1\varphi(\eta^{n+1})\}$. Then for $i \geq j+2$ we have

$$\begin{aligned} P_x^\omega(A_i^\omega \cap A_j^\omega) &= P_x^\omega(\sigma_j \leq \eta^{j+1}, \sigma_i \leq \eta^{i+1}) \\ &= E_x^\omega \left[1_{\{\sigma_j \leq \eta^{j+1}\}} P_{Y_{\sigma_j}^\omega}^\omega (d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t \in (\eta^i - \sigma_j, \eta^{i+1} - \sigma_j]) \right] \\ &\leq E_x^\omega \left[1_{\{\sigma_j \leq \eta^{j+1}\}} P_{Y_{\sigma_j}^\omega}^\omega (d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}) \right] \\ &\leq \left(\sup_{z: d(x, z) \leq 2c_1\varphi(\eta^{j+1})} P_z^\omega (d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}) \right) \cdot P_x^\omega(\sigma_j \leq \eta^{j+1}). \end{aligned} \quad (4.8)$$

By Lemma 2.12, for any $i \geq j+2$ with

$$\eta^i - \eta^{j+1} \geq (c_1\varphi(\eta^{i+1}))^\beta, \quad 2c_1\varphi(\eta^{j+1}) \leq c_1\varphi(\eta^{i+1}), \quad \varphi(\eta^{i+1}) \geq \max_{z \in B(x, c_1\varphi(\eta^{i+1}))} N_z(\omega) \quad (4.9)$$

we have

$$\begin{aligned} &\left(\sup_{z: d(x, z) \leq 2c_1\varphi(\eta^{j+1})} P_z^\omega (d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t > \eta^i - \eta^{j+1}) \right) \\ &\leq \frac{c_3 (c_1\varphi(\eta^{i+1}))^{\alpha-\beta} (\eta^i - \eta^{j+1})}{(\eta^i - \eta^{j+1})^{\alpha/\beta}} \leq \frac{c_4 (c_1\varphi(\eta^{i+1}))^{\alpha-\beta} \eta^i}{(\eta^i)^{\alpha/\beta}}. \end{aligned} \quad (4.10)$$

(4.9) holds for sufficiently large i, j with $i \geq j+2$ (say $j \geq N_4 = N_4(\omega)$) by (4.5), (4.6), Assumption 1.3 (3) and Lemma 2.15 (3). By Lemma 2.14, for any i with

$$\eta^i \geq T_0, \text{ where } T_0 \text{ is as in Lemma 2.14, } \eta^i \geq (c_1\varphi(\eta^{i+1}))^\beta, \quad c_1\varphi(\eta^{i+1}) \geq \max_{v \in B(x, c_1\eta^{i/\beta})} N_v(\omega) \quad (4.11)$$

we have

$$\begin{aligned} &\frac{(c_1\varphi(\eta^{i+1}))^{\alpha-\beta} \eta^i}{(\eta^i)^{\alpha/\beta}} \leq c_6 P_x^\omega (d(x, Y_t^\omega) \leq 2c_1\varphi(\eta^{i+1}) \text{ for some } t \in (\eta^i, \eta^{i+1}]) \\ &= c_6 P_x^\omega(A_i^\omega). \end{aligned} \quad (4.12)$$

(4.11) holds for sufficiently large j (say $j \geq N_5 = N_5(\omega)$) by (4.5), Assumption 1.3 (3) and Lemma 2.15 (3). Hence by (4.8), (4.10) and (4.12) we have $P_x^\omega(A_i^\omega \cap A_j^\omega) \leq cP_x^\omega(A_i^\omega)P_x^\omega(A_j^\omega)$ for sufficiently large j ($j \geq N_6 := N_4 \vee N_5$) and $i \geq j + 2$. In the case of $i = j + 1$ we have $P_x^\omega(A_{j+1}^\omega \cap A_j^\omega) \leq P_x^\omega(A_j^\omega)$. Thus we obtain the condition (3) of Lemma 4.3 for $\{A_i^\omega\}_{i \geq N_6}$.

By Lemma 4.3, we thus complete the proof. \square

By Theorem 4.1 and Theorem 4.2 we complete the proof of Theorem 1.5.

5 Ergodic media

In this section, we consider the case $G = (V, E) = \mathbb{Z}^d$ and obtain Theorem 1.7 under Assumption 1.6. We follow the strategy as in [15]

5.1 Ergodicity of the shift operator on $\Omega^\mathbb{Z}$

We consider Markov chains on the random environment, which is called the environment seen from the particle, according to Kipnis and Varadhan [20].

Let $\Omega = [0, \infty)^E$ and define \mathcal{B} as the natural σ -algebra (generated by coordinate maps). We write $\mathcal{Y} = \Omega^\mathbb{Z}$, $\mathcal{Y} = \mathcal{B}^{\otimes \mathbb{Z}}$. If each conductance may take the value 0, we regard 0 as the base point and define $\mathcal{C}_0(\omega) = \{x \in \mathbb{Z}^d \mid 0 \xleftrightarrow{\omega} x\} = V(G^\omega)$, where $0 \xleftrightarrow{\omega} x$ means that there exists a path $\gamma = e_1 e_2 \cdots e_k$ from 0 to x such that $\omega(e_i) > 0$ for all $i = 1, 2, \dots, k$. Define $\Omega_0 = \{\omega \in \Omega \mid \sharp \mathcal{C}_0(\omega) = \infty\}$ and $\mathbb{P}_0 = \mathbb{P}(\cdot \mid \Omega_0)$.

Next we consider the Markov chains seen from the particle. Recall that $\{X_n^\omega\}_{n \geq 0}$ is the discrete time random walk which is introduced in Section 1.1. Let $\omega_n(\cdot) = \omega(\cdot + X_n^\omega) = \tau_{X_n^\omega} \omega(\cdot) \in \Omega$. We can regard this Markov chain $\{\omega_n\}_{n \geq 0}$ as being defined on $\mathcal{Y} = \Omega^\mathbb{Z}$. We define a probability kernel $Q : \Omega_0 \times \mathcal{B} \rightarrow [0, 1]$ as

$$Q(\omega, A) = \frac{1}{\sum_{e': |e'|=1} \omega_{e'}} \sum_{v: |v|=1} \omega_{0v} 1_{\{\tau_v \omega \in A\}}.$$

This is nothing but the transition probability of the Markov chain $\{\omega_n\}_{n \geq 0}$.

Next we define the probability measure on $(\mathcal{Y}, \mathcal{Y})$ as

$$\mu((\omega_{-n}, \dots, \omega_n) \in B) = \int_B \mathbb{P}_0(d\omega_{-n}) Q(\omega_{-n}, d\omega_{-n+1}) \cdots Q(\omega_{n-1}, d\omega_n).$$

By the above definition, $\{\tau_{X_k^\omega} \omega\}_{k \geq 0}$ has the same law in $\mathbb{E}_0(P_0^\omega(\cdot))$ as $(\omega_0, \omega_1, \dots)$ has in μ , that is,

$$\mathbb{E}_0[P_0^\omega(\{\tau_{X_k^\omega} \omega\}_{k \geq 0} \in B)] = \mu((\omega_0, \omega_1, \dots) \in B) \quad (5.1)$$

for any $B \in \mathcal{Y}$.

We need the following Theorem. Let $T : \mathcal{Y} \rightarrow \mathcal{Y}$ be a shift operator of \mathcal{Y} , that is,

$$(T\omega)_n = \omega_{n+1}.$$

Theorem 5.1. *Under Assumption 1.6, T is ergodic with respect to μ .*

The proof is similar to [8, Proposition 3.5], so we omit it.

5.2 The Zero-One law

The purpose of this subsection is to give the Zero-One law (see Proposition 5.2). Let $a \geq 0$ and $A_1^\omega(a), A_2^\omega(a), A_3^\omega(a)$ be the events

$$\begin{aligned} A_1^\omega(a) &= \left\{ \limsup_{n \rightarrow \infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta} (\log \log n)^{1-1/\beta}} > a \right\}, \\ A_2^\omega(a) &= \left\{ \limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta} (\log \log n)^{1-1/\beta}} > a \right\}, \\ A_3^\omega(a) &= \left\{ \liminf_{n \rightarrow \infty} \frac{\max_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta} (\log \log n)^{1-1/\beta}} > a \right\}. \end{aligned}$$

Define

$$\tilde{A}_i(a) = \{\omega \in \Omega \mid A_i^\omega(a) \text{ holds for } P_x^\omega\text{-a.s. and for all } x \in \mathcal{C}_0(\omega)\}.$$

Proposition 5.2. $\mathbb{P}_0(\tilde{A}_i(a))$ is either 0 or 1.

Proof. See [23, Proposition 5.2]. □

5.3 Proof of Theorem 1.7

In this subsection we discuss the proof of Theorem 1.7. Recall $T_0^\omega = 0$, $T_{n+1}^\omega = \inf\{t > T_n^\omega \mid Y_t^\omega \neq Y_{T_n^\omega}^\omega\}$ and $X_n^\omega = Y_{T_n^\omega}^\omega$.

First we consider the CSRW. $\{T_{n+1}^\omega - T_n^\omega\}_{n \geq 0}$ is a family of i.i.d. random variables whose distributions are exponential with mean 1, so the law of large number gives us

$$\frac{T_n^\omega}{n} \rightarrow 1 \quad P_0^\omega\text{-a.s.}$$

Thus

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \limsup_{n \rightarrow \infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}, \\ \limsup_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}, \\ \liminf_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \liminf_{n \rightarrow \infty} \frac{\sup_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}. \end{aligned}$$

By Assumption 1.6, Proposition and Theorem 1.4 we obtain Theorem 1.7.

Next we consider the VSRW. $\{T_{n+1}^\omega - T_n^\omega\}_{n \geq 0}$ are non-i.i.d., and the distribution of $T_{n+1}^\omega - T_n^\omega$ is exponential with mean $\frac{1}{\pi^\omega(X_n^\omega)}$. Write S_x^ω be a exponential random variable with parameter $\pi^\omega(x)$ and $\bar{S}_x(\bar{\omega}) := S_x^{\bar{\omega}_0}$, ($\bar{\omega} \in \mathcal{Y}$). Then by (5.1) and the ergodicity we have

$$\begin{aligned} \frac{1}{n} T_n^\omega &= \frac{1}{n} \sum_{k=0}^{n-1} S_{X_k^\omega}^\omega \stackrel{d}{=} \frac{1}{n} \sum_{k=0}^{n-1} \bar{S}_0(T^k \bar{\omega}) \rightarrow \mathbb{E}^\mu [\bar{S}_0] \\ &= \mathbb{E} [E_0^\omega [S_0^\omega]] = \int_\Omega \int_0^\infty x \pi^\omega(0) \exp(-\pi^\omega(0)x) dx d\mathbb{P} = \mathbb{E} \left[\frac{1}{\pi^\omega(0)} \right]. \end{aligned}$$

Thus

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{d(Y_0^\omega, Y_t^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \left(\frac{1}{\mathbb{E} \left[\frac{1}{\pi^\omega(0)} \right]} \right)^{1/\beta} \limsup_{n \rightarrow \infty} \frac{d(X_0^\omega, X_n^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}, \\ \limsup_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \left(\frac{1}{\mathbb{E} \left[\frac{1}{\pi^\omega(0)} \right]} \right)^{1/\beta} \limsup_{n \rightarrow \infty} \frac{\sup_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}, \\ \liminf_{t \rightarrow \infty} \frac{\sup_{0 \leq s \leq t} d(Y_0^\omega, Y_s^\omega)}{t^{1/\beta}(\log \log t)^{1-1/\beta}} &= \left(\frac{1}{\mathbb{E} \left[\frac{1}{\pi^\omega(0)} \right]} \right)^{1/\beta} \liminf_{t \rightarrow \infty} \frac{\sup_{0 \leq k \leq n} d(X_0^\omega, X_k^\omega)}{n^{1/\beta}(\log \log n)^{1-1/\beta}}. \end{aligned}$$

By Assumption 1.6, Proposition 5.2 and Theorem 1.4 we obtain Theorem 1.7.

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